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Loop shaping design related to LQG/LTR for SISO minimum phase plants

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One method of model-based compensator design for linear systems consists of two stages: state feedback design and observer design. A key issue in recent work in multivariable synthesis involves selecting the observer (state feedback) gain so that the final loop transfer function is the same as the state feedback (observer) loop transfer function. This is called loop transfer recovery (LTR) (Athans and Stein 1987, Kazerooni and Houpt 1986, Kazerooni *et al.* 1985, Doyle and Stein 1981). This paper shows how identification of the internal mechanism of the LTR provides simple design rules with little algebra for single-input single-output (SISO) systems. In the SISO case, the LQG/LTR reduces to computation of a compensator that shapes the loop transfer function by (i) cancelling the zeros of the plant with the compensator poles, and (ii) locating a new set of zeros for the compensator to shape the loop transfer function.

Nomenclature

A, B and C	plant parameters
$D(t)$	disturbance signal
$E(t)$	error signal
$E_m(s)$	modelling error
$G_p(s)$	nominal transfer function of the plant
$G_a(s)$	actual transfer function of the plant
G	state feedback gain, $1 \times n$
H	observer gain, $n \times 1$
K_0	compensator DC gain
$K(s)$	compensator transfer function
$N(t)$	noise signal
n	order of the plant
$P(s)$	pre-compensator
p_i	poles of the compensator
$R(t)$	input command to the system
u_i	right eigenvector of the state feedback design
v_i	left eigenvector of the observer
$X(t)$, $Y(t)$, $U(t)$	states, output and input of the plant
$\hat{X}(t)$, $\hat{Y}(t)$	states and output of the observer
$Y_a(t)$	actual output of the plant
z_i	zeros of the compensator
λ_i	eigenvalues of the state feedback configuration
μ_i	eigenvalues of the observer
ε_R	tracking error

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ε_D output deviation in response to disturbances
ε_{MD} output deviation due to modelling error
ε_N output deviation in response to noise signal
ω_R frequency range of the input command
ω_D frequency range of the disturbances
ω_{MD} frequency range of the modelling error
ω_N frequency range of noise signal
ρ a positive scalar

1. Introduction

Multivariable control systems can hardly be designed without the use of modern control theory and its underlying computer algorithms. Classical control theory could be applied to a bulk of lower order SISO industrial systems. Hence, much of modern control theory has had little impact on the design of the most common types of control systems. This paper is concerned with the design of compensators for SISO systems, using the underlying principles of the modern control theory concepts of LQG/LTR.

The modern control theory has something definite to say about improving the design techniques for common control systems, but the mathematical nature of the theory has proven to be a detriment to its widespread use. This paper presents an 'almost' systematic design methodology with little algebra and some rules of thumb for designing compensators for lower order SISO systems with left half-plane zeros (minimum phase). An engineer with experience, who does not want to get involved with matrix differential equations, quadratic integral performance index and other mathematical details of modern control theory should be able to follow this methodology.

The work presented here is a frequency domain approach for compensator design for SISO systems. We start with presenting a set of practical design specifications. Establishing the set of design specifications gives the designers a chance to express what they wish to happen for the controlled system. Although this set of performance specifications does not imply any choice of control method, the control technique presented here is a natural consequence of the way the design specifications are formulated. Such a set allows the designers to translate their objectives into a form that is meaningful from the standpoint of control theory. This set of performance specifications is a contemporary and practical way of formulating the properties that will enable the closed-loop system to operate according to the designer's choice.

2. SISO systems

We will deal with standard feedback configuration as shown in Fig. 1. It consists of a plant $G_p(s)$, controller $K(s)$ forced by command $R(s)$, measurement noise $N(s)$ and

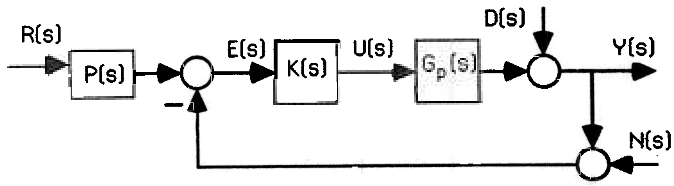

Figure Standard feedback configuration.

disturbances $D(s)$. All disturbances are assumed to be reflected at the output of the plant. Note that all disturbances that arrive in the loop at the input to the plant can always be reflected at the output to the plant by proper dynamic scaling of the disturbance. The optional pre-compensator, $P(s)$ is used to calibrate the input command. Both nominal mathematical models for the plant and the controller are rational transfer functions.

The dynamic behaviour of the plant, $G_p(s)$, is modelled by the linear time-invariant system as

$$\dot{X} = AX + BU, \quad Y = CX, \quad \text{with} \quad X \in \mathbb{R}^n, Y \text{ and } U \in \mathbb{R}^1 \tag{1}$$

where

$$G_p(s) = C(sI - A)^{-1}B \quad \text{and} \quad s = j\omega$$

For SISO systems, the compensator $K(s)$ is considered of the form given by (2)

$$K(s) = K_0 \frac{(s/z_1 + 1)(s/z_2 + 1) \dots (s/z_m + 1)}{(s/p_1 + 1)(s/p_2 + 1) \dots (s/p_n + 1)}, \quad n > m \tag{2}$$

3. Design specifications

We propose to design the compensator $K(s)$, for SISO systems using the principles of LQG/LTR, such that the closed-loop system shown in Fig. 1 is stable and satisfies the following five design specifications in frequency domain.

(i) Tracking of the input command which is bounded in magnitude and frequency very 'closely' for all the frequency range of the input. We define the closeness of the system output to input command by the following inequality

$$\frac{|Y(j\omega) - R(j\omega)|}{|R(j\omega)|} < \varepsilon_R, \quad \text{for all} \quad \omega \in \omega_R \tag{3}$$

where ω_R is the frequency range of input. Note that ε_R (the tracking error), expressed by the designer, is a small number that shows the closeness of the output to input (e.g. for good tracking systems ε_R could be down to 0·05).

(ii) Rejection of all disturbances $D(j\omega)$ that are bounded in magnitude and frequency. By 'rejection' we mean

$$\frac{|Y(j\omega)|}{|D(j\omega)|} < \varepsilon_D, \quad \text{for all} \quad \omega \in \omega_D \tag{4}$$

where ω_D is the frequency range of the disturbances, ε_D, a small scalar expressed by designer, represents the compliancy of the system.

(iii) Performance robustness to bounded modelling errors. The model uncertainties fall into two classes. Lack of exact knowledge about the parameters of the modelled dynamics constitute the first class of model uncertainties. High frequency unmodelled dynamics form the second class of model uncertainties. Here we deal with the first class of uncertainties. The second class of uncertainties are discussed in item (iv). One of the primary purposes of using feedback in control systems is to reduce the performance sensitivity of the system to parameter variations of the plant. The parameters of a system may vary with age, with changing environment (e.g. ambient temperature), due to changes in the parameters of the hardware of the controller. Also modelling errors from numerical round-off errors induced by the digital computer may exist, while constructing the mathematical model of the controller. Conceptually, sensitivity to modelling errors is a measure of the effectiveness of feedback in reducing the influence of parameter variations on system performance. If the bounded modelling error in the system is $\delta G_p(j\omega)$, then the output for a given input $R(j\omega)$ will be

$$Y(j\omega) + \delta Y(j\omega) = \frac{[G_p(j\omega) + \delta G_p(j\omega)]K(j\omega)}{1 + [G_p(j\omega) + \delta G_p(j\omega)]K(j\omega)} R(j\omega) \tag{5}$$

where $[G_p(j\omega) + \delta G_p(j\omega)]$ and $[Y(j\omega) + \delta Y(j\omega)]$ are the true model and the true output of the plant, respectively. A system has performance robustness if the ratio of the deviation of the system output $\delta Y(j\omega)$ to the nominal output $Y(j\omega)$ is 'small'. To provide performance robustness to modelling error, one would like to guarantee the following inequality for the system

$$\frac{|\delta Y(j\omega)|}{|Y(j\omega)|} < \varepsilon_{MD}, \quad \text{for all} \quad \omega \in \omega_{MD} \tag{6}$$

where ω_{MD} is the frequency range in which the modelling error $\delta G_p(j\omega)$ occurs.

(iv) Stability robustness to bounded unmodelled dynamics. One can have errors from several sources. These include: intentional approximation of higher order dynamics by lower order models; neglecting fast actuator and sensor dynamics; neglecting some or all bending and torsional dynamics; ignoring far away poles, minimum and non-minimum phase zeros; and small time delays. These uncertainties in the plant can drive a nominally stable system into instability. Hence the compensator has to make the system robust to unstructured uncertainties in the plant. We deal with this robustness via Nyquist's criterion.

(v) Output insensitivity to noise at higher frequencies. By 'insensitivity to noise' we mean that the output $Y(j\omega)$ is not polluted by noise $N(j\omega)$ or

$$\frac{|Y(j\omega)|}{|N(j\omega)|} < \varepsilon_N, \quad \text{for all} \quad \omega \in \omega_N \tag{7}$$

where ω_N is the frequency range of noise. ε_N is a small scaler specified by the engineer and represents the allowable fluctuation of the output in response to measurement noise.

The above design specifications do not imply any design method; they only allow the designers to express what they want to have in the system in a form that is meaningful from the stand point of the control theory. One must translate the above design specifications into mathematical terms. Referring to Fig. 1 the nominal output of the plant $Y(s)$ and the error signal $E(s)$, are

$$Y(s) = \frac{G_p(s)K(s)}{1 + G_p(s)K(s)} R(s) + \frac{1}{1 + G_p(s)K(s)} D(s) - \frac{G_p(s)K(s)}{1 + G_p(s)K(s)} N(s) \tag{8}$$

$$E(s) = \frac{1}{1 + G_p(s)K(s)} R(s) + \frac{1}{1 + G_p(s)K(s)} D(s) - \frac{1}{1 + G_p(s)K(s)} N(s) \tag{9}$$

On examining the above two equations, the five design specifications can mathematically be expressed as inequality constraints on the loop transfer function $G_p(j\omega)K(j\omega)$.

(i) Considering (9), inequality (10) must be satisfied to guarantee that the output follows the input command with tracking error of ε_R.

$$\frac{}{|1 + G_p(j\omega)K(j\omega)|} < \varepsilon_R, \quad \text{for all} \quad \omega \in \omega_R \tag{10}$$

A more conservative bound is given by inequality (11).

$$|G_p(j\omega)K(j\omega)| > 1 + \frac{1}{\varepsilon_R}, \quad \text{for all} \quad \omega \in \omega_R \tag{11}$$

Since for a good tracking system ε_R is a small number, therefore the loop transfer function, $|G_p(j\omega)K(j\omega)|$ must be very large (inequality (11)) for all $\omega \in \omega_R$ to guarantee the closeness of the output to input.

(ii) For rejection of disturbances $D(j\omega)$ as expressed by inequality (4), inequality (12) must be satisfied.

$$\frac{}{|1 + G_p(j\omega)K(j\omega)|} < \varepsilon_D, \quad \text{for all} \quad \omega \in \omega_D$$

A more conservative bound is given by inequality (13).

$$|G_p(j\omega)K(j\omega)| > 1 + \frac{}{\varepsilon_D}, \quad \text{for all} \quad \omega \in \omega_D$$

Inequality (13) means that the loop transfer function, $|G_p(j\omega)K(j\omega)|$, must be very large for all the frequency range of the disturbances (i.e. for all $\omega \in \omega_D$) to guarantee the rejection of the disturbance.

(iii) $G_p(s)$ does not express the true dynamic behaviour of the plant. The actual plant dynamic $G_a(j\omega)$ could be written as $G_a(j\omega) = G_p(j\omega) + \delta G_p(j\omega)$, where $\delta G_p(j\omega)$ is modelling error. The compensator has to take into account the errors due to modelling errors to ensure the performance. The output error due to modelling is given by $\delta Y(j\omega) = Y_a(j\omega) - Y(j\omega)$, where $Y_a(j\omega)$ is the actual output of the plant, in the frequency range we have modelled the plant. We call this frequency range ω_{MD}. After some algebra

$$\frac{|\delta Y(j\omega)|}{|Y(j\omega)|} = \frac{1}{|1 + G_a(j\omega)K(j\omega)|} \times \frac{|\delta G_p(j\omega)|}{|G_p(j\omega)|} \tag{14}$$

To reduce the error from modelling, $|G_a(j\omega)K(j\omega)|$ has to be very large for all the frequency range where the modelling error of the plant occurs.

$$|1 + G_a(j\omega)K(j\omega)| > \frac{|\delta G_p(j\omega)|}{|G_p(j\omega)|\varepsilon_{MD}}, \quad \text{for all } \omega \in \omega_{MD} \tag{15}$$

Since $|G_p(j\omega)| \gg |\delta G_p(j\omega)|$, from the above inequality, the sensitivity transfer function $|1 + G_p(j\omega)K(j\omega)|$ must guarantee the following inequality for performance robustness.

$$|1 + G_p(j\omega)K(j\omega)| > \frac{|\delta G_p(j\omega)|}{|G_p(j\omega)|\varepsilon_{MD}}, \quad \text{for all} \quad \omega \in \omega_{MD} \tag{16}$$

A more conservative bound is given by inequality (17).

$$|G_p(j\omega)K(j\omega)| > \frac{|\delta G_p(j\omega)|}{|G_p(j\omega)|\varepsilon_{MD}} + 1, \quad \text{for all} \quad \omega \in \omega_{MD} \tag{17}$$

For good performance robustness (small ε_{MD}), $|G_p(j\omega)K(j\omega)|$ must be chosen as a large number.

(iv) Stability robustness: suppose $G_a(j\omega)$ represents the true dynamics of the plant that contains all the unmodelled modes. Figure 2 shows $G_a(j\omega)$ and the nominal model $G_p(j\omega)$ of the plant for some frequency range. Practitioners always observe the high frequency modes in experimentally derived frequency response of the plant.

Let $G_a(j\omega) = G_p(j\omega)[1 + E_m(j\omega)]$, where E_m is the multiplicative error due to unmodelled dynamics. $E_m(j\omega)$ is the best educated guess of the error on evaluation of the true dynamic of the system. One could determine a conservative bound for the error function by taking measurements and comparing the actual and nominal outputs.

We use Nyquist's criteria to guarantee the stability of the closed-loop system in the presence of unmodelled dynamics. Suppose the nominal system is closed-loop stable. By Nyquist's criteria (Fig. 3), for stability of the closed-loop system in the presence of unmodelled dynamics the following inequality must be satisfied.

$$|G_a(j\omega)K(j\omega) - G_p(j\omega)K(j\omega)| < |d(j\omega)|, \quad \text{for all} \quad 0 < \omega < \infty \tag{18}$$

where $G_p(j\omega)K(j\omega) = -1 + d(j\omega)$ (Fig. 3). Note that inequality (18) is valid even when the system is unstable and the count around -1 must be considered for stability analysis (Lehtomaki *et al.* 1975). Inequality (18) states that the deviation of the loop transfer function (because of the uncertainties) must be such that the actual loop transfer function stays away from point -1 for all $0 < \omega < \infty$. If any part of the loop transfer function deviates from its nominal value such that it passes point -1, then the counts around point -1 may change and stability may not be guaranteed. This

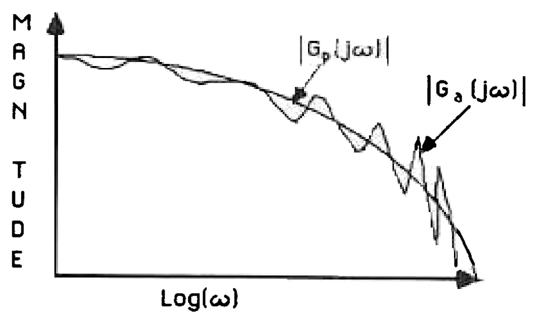

Figure 2. Actual and nominal plant dynamics.

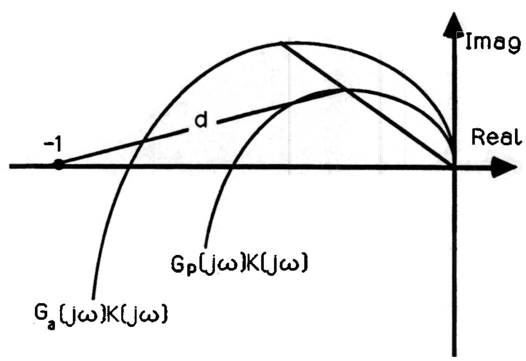

Figure 3. Nyquist's criterion.

concept is extensively described by Lehtomaki *et al.* (1975). Substituting for $d(j\omega)$ in inequality (18)

$$|G_a(j\omega)K(j\omega) - G_p(j\omega)K(j\omega)| < |1 + G_p(j\omega)K(j\omega)|, \quad \text{for all} \quad 0 < \omega < \infty \quad (19)$$

Considering $G_a(j\omega) = G_p(j\omega)[1 + E_m(j\omega)]$ inequality (19) can be written as

$$\frac{|G_p(j\omega)K(j\omega)|}{|1 + G_p(j\omega)K(j\omega)|} < \frac{1}{|E_m(j\omega)|}, \quad \text{for all} \quad 0 < \omega < \infty \quad (20)$$

At high frequencies approximation (21) is true

$$\frac{|G_p(j\omega)K(j\omega)|}{|1 + G_p(j\omega)K(j\omega)|} \approx |G_p(j\omega)K(j\omega)| \quad (21)$$

Hence for stability robustness to high frequency unmodelled dynamics, we should guarantee inequality (22) at high frequencies (Lehtomaki *et al.* 1981).

$$|G_p(j\omega)K(j\omega)| < \frac{1}{|E_m(j\omega)|} \quad \text{(only at high frequencies)} \quad (22)$$

(v) For insensitivity to noise as stated by inequality (7), inequality (23) must be guaranteed.

$$\frac{|G_p(j\omega)K(j\omega)|}{|1 + G_p(j\omega)K(j\omega)|} < \varepsilon_N, \quad \text{for all} \quad \omega \in \omega_N \quad (23)$$

or

$$|G_p(j\omega)K(j\omega)| < \frac{\varepsilon_N}{1 + \varepsilon_N}, \quad \text{for all} \quad \omega \in \omega_N \quad (24)$$

Inequality (24) simply states that the loop transfer function $|G_p(j\omega)K(j\omega)|$ must be very small for all $\omega \in \omega_N$ to guarantee the insensitivity of the output to the noise (ε_N is usually chosen as a very small number).

The design specifications stated above on the loop transfer function $|G_p(j\omega)K(j\omega)|$ are pictured graphically in Fig. 4. Note that the stability robustness to high frequency unmodelled dynamics requires small gain for the loop gain at high frequencies while the performance robustness to modelling error, tracking and disturbance rejection require large loop gain at low frequencies. The loop gain must be shaped to cross over before the frequency range of unmodelled dynamics. No rules are given for the shape

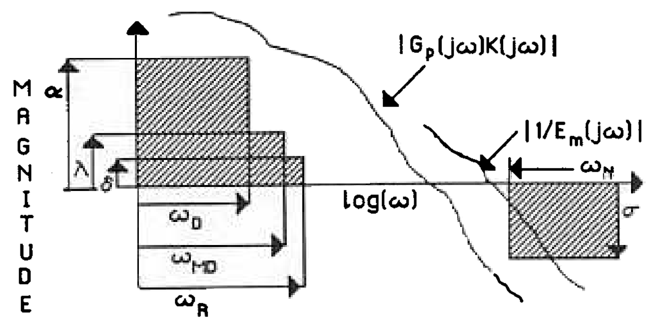

Figure 4. Design specifications on $|G_p(j\omega)K(j\omega)|$.

of the loop transfer function at mid-frequencies (around the neighbourhood of the cross-over frequency). The cross-over frequency and the phase margin have historically been used to investigate the relative stability. Since this method investigates the relative stability by the direct analysis of the Nyquist criteria, it does not need to consider the phase margin as a tool for stability criteria. Note that inequality (18) establishes a conservative bound for the error. This bound guarantees the stability of the closed-loop system in the presence of uncertainties in the magnitude and regardless of the phase in the plant.

If the compensated system does not satisfy the stability robustness specifications, the system may become unstable. If one cannot meet the stability robustness specifications at high frequencies by clearing off the design specification of Fig. 4, it is necessary to consider the higher-order dynamics (if at all possible) when modelling the system. Adding the higher-order dynamics to the system allows for weaker stability robustness specifications at higher frequencies. If higher-order dynamics cannot be determined, it is necessary to compromise on values of ε_R and ε_D (larger ε_R and ε_D or smaller ω_R and ω_D). The small frequency range of tracking and disturbance signals will allow designers to meet strong sets of stability robustness specifications at high frequencies. To achieve a wide range of tracking and disturbances, designers should have a good model of the plant at high frequencies and consequently, a weak set of stability robustness specifications at high frequencies. Because of the conflict between the desired frequency range of tracking signals (and also disturbances) and stability robustness to high frequency dynamics, it is a struggle to meet both sets of specifications for a given modelled uncertainty. The frequency range of tracking cannot be selected to be arbitrarily wide if a good model of the plant does not exist at high frequencies, while a good model of the plant at high frequencies makes it possible to retain good disturbance rejection and tracking (small ε_R and ε_D) for a wide frequency range (large ω_R and ω_D).

4. Design procedure

The objective is to design a compensator, $K(s)$ such that $G_p(s)K(s)$ passes through the constraints given in Fig. 4. One traditional method of designing $K(s)$ consists of two stages. The first stage concerns state-feedback gain design (Klein and Moore 1977). A state-feedback gain G is designed so that the closed-loop system in Fig. 5 is stable or equivalently

$$\left.\begin{array}{ll}(\lambda_i I_n - A + BG)u_i = 0_n, & i = 1, 2, \ldots, n \\ \text{real}\,(\lambda_i) < 0, & u_i \neq 0\end{array}\right\} \tag{25}$$

where λ_i and u_i ($i = 1, 2, \ldots, n$) are closed-loop eigenvalues and eigenvectors of state feedback configuration.

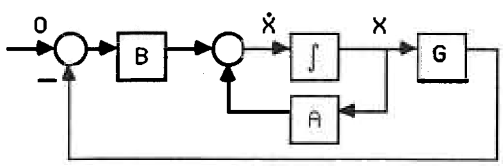

Figure 5. State feedback configuration.

In the second stage, an observer gain H is designed to make the first stage realizable (Luenberger 1971). Figure 6 shows the structure of the observer. For stability of the observer the following equality must be guaranteed.

$$\left.\begin{aligned} v_i^{\mathrm{T}}(\mu_i I_n - A + HC) &= 0_n^{\mathrm{T}}, \quad i = 1, 2, \ldots, n \\ \text{real}\,(\mu_i) &< 0, \qquad\qquad v_i^{\mathrm{T}} \neq 0 \end{aligned}\right\} \tag{26}$$

where μ_i and v_i^{T} $(i = 1, 2, \ldots, n)$ are the closed-loop eigenvalues and left eigenvectors of the observer. Combining the state feedback and observer designs (Fig. 7) yields the unique compensator transfer function matrix given by

$$K(s) = G(sI - A + BG + HC)^{-1}H \tag{27}$$

The closed-loop system shown in Fig. 7 is stable iff the loops in Figs. 5 and 6 are stable (separation principle). In this paper, a different approach in design of H and G is taken. First, one designs a stabilizing H (stabilizing H implies an H that $A - HC$ has eigenvalues in the left half-plane) such that the loop transfer function $C(sI - A)^{-1}H$ in Fig. 6 meets the frequency-dependent design specifications. Later we mention how to design a stabilizing H such that the design specifications are satisfied in frequency domain. In the second stage of the compensator design, a stabilizing state feedback gain G is designed to guarantee that the final loop transfer function $G_p(s)K(s)$ maintains the same loop shape that $C(sI - A)^{-1}H$ achieved via filter design at the first stage. This is the principle behind loop transfer recovery (Athans and Stein 1987, Kazerooni and Houpt 1986).

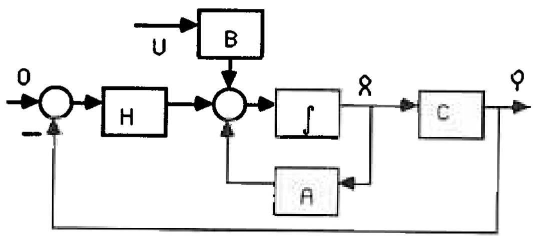

Figure 6. Observer configuration.

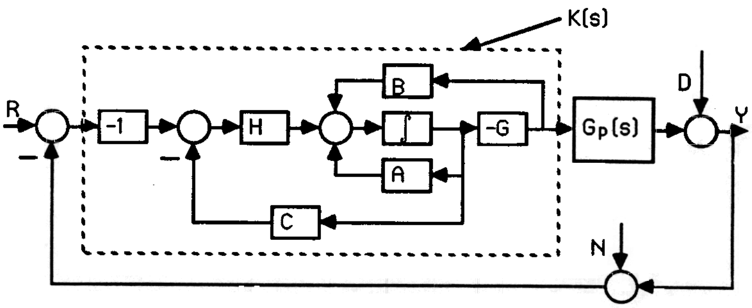

Figure 7. Closed-loop system.

Summary of eigenstructural properties of LTR

Historically, the LTR method is the consequence of attempts by Doyle and Stein (1979, 1981) to improve the robustness of linear quadratic gaussian (LQG) regulators.

In their seminal work, Doyle and Stein address the problem of finding the steady state observer gain that assures the recovery of the loop transfer function resulting from full state feedback. First, they demonstrate a key lemma that gives a sufficient condition for the steady-state observer gain such that LTR takes place. To compute the gain, they show that the infinite time-horizon Kalman filter formulation with 'small' white measurement noise covariance yields an observer gain that satisfies the sufficient condition for the loop-transfer recovery.

The eigenstructure properties of LTR for a general multi-input multi-output system has been discussed by Kazerooni and Houpt (1986) and Kazerooni *et al.* (1985). Some of those properties for SISO systems are listed here.

(i) If G is chosen such that limit (28) is true as ρ approaches zero for any non-singular $(m \times m)$ W matrix

$$\sqrt{\rho} G \rightarrow WC$$

then $K(s)$ approaches pointwise (non-uniformly) towards expression (29)

$$[C(sI - A)^{-1}B)]^{-1}C(sI - A)^{-1}H \tag{29}$$

and since $G_p(s) = C(sI - A)^{-1}B$, then $G_p(s)K(s)$ will approach $C(sI - A)^{-1}H$ non-uniformly.

(ii) The finite zeros (Davison and Wang 1974) of the compensator $K(s)$ are the same as the finite zeros of $C(sI - A)^{-1}H$.

(iii) If ρ approaches zero then all the eigenvalues of the compensator $K(s)$, approach the zeros (including ones at infinity) of the plant.

Since the number of zeros of two cascaded systems [$G_p(s)$ and $K(s)$] is the sum of number of zeros of both systems in the limit, the zeros of $G_p(s)K(s)$ after cancellation are the same as zeros of $K(s)$ or $C(sI - A)^{-1}H$. Similar arguments can be given for the poles of $G_p(s)K(s)$. The poles of $K(s)$ cancel out with the zeros of the plant. Therefore the poles of $G_p(s)K(s)$ will be the same as the poles of $G_p(s)$ or $C(sI - A)^{-1}H$. The above comment concerning pole-zero cancellation explains the eigenstructure mechanism for LTR. This argument does not prove the equality of $G_p(s)K(s)$ and $C(sI - A)^{-1}H$ as ρ approaches zero. In summary, the poles of $K(S)$ approach the zeros of the plant and the zeros of $K(s)$ approach the zeros of $C(sI - A)^{-1}H$. The cancellation mechanism in the multi-input multi-output systems has been explained in detail by Kazerooni and Houpt (1986). We are planning to design a $K(s)$ of the structure given by (2).

According to property (iii), $p_1, p_2, \ldots, p_n$ must be equal to zeros of the plant; therefore, the zeros of $G_p(S)K(S)$ will be equal to $z_1, z_2, \ldots, z_m$ only. The design centres on locating the zeros of $K(s)$ on appropriate locations such that $G_p(s)K(s)$ has the same zeros as $C(sI - A)^{-1}H$. Once the zeros of $K(s)$ are located, the poles of $K(s)$ will be chosen equal to zeros of $G_p(s)$ to make $K(s)$ a rational transfer function. In choosing the zeros of $K(s)$, we consider the following two cases of stable and unstable systems with left half-plane zeros.

5. Examples

The following two examples depict the design method for minimum phase plants.

Example 1

Consider a stable plant whose dynamics are given by the following rational transfer function

$$G_p(s) = 10\frac{(s + 1)}{(s/5 + 1)(10s + 1)} \tag{30}$$

It is desired to design a compensator such that the plant satisfies the design specifications with respect to Fig. 4. In brief they are: $\omega_R = 10$ rad/s; $\omega_D = 4$ rad/s; $\omega_N = 200$ rad/s; $\delta = 10$ dB; $\alpha = 20$ dB; $\sigma = 10$ dB. $|E_m(j\omega)|$ is given in Fig. 8.

Since the zeros of $G_p(s)$ will be cancelled out by the poles of the compensator $K(s)$, the compensated loop transfer function $G_p(s)K(s)$ at low frequencies consists of the zeros of the $K(s)$ and the poles of $G_p(s)$. We are looking for a set of zeros [zeros of $K(s)$] such that the transfer function consisting of this set of zeros (maximum of $n - 1$ finite zeros) and the poles of the $G_p(s)$ clears off the design specifications while the stability is guaranteed. Figure 8 shows the magnitude and phase of the denominator of $G_p(s)$ with unity gain given by (31)

$$\frac{}{(s/5 + 1)(10s + 1)} \tag{31}$$

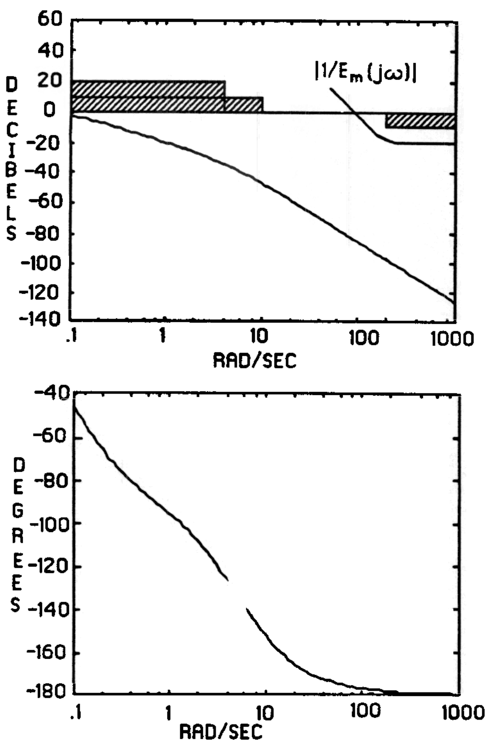

Figure 8. Gain and phase plots of the transfer function given by (31).

The design procedure centres around location of one zero ($n - 1 = 1$) to clear off the design specifications while the loop is stable. We choose the zero at -10 to guarantee the stability of the system. We also choose a loop gain of 563·64 such that the loop passes through the design specifications. Note that the location of this zero is somewhat arbitrary. Figure 9 shows the plot of (32)

$$563{\cdot}64\frac{(s/10 + 1)}{(s/5 + 1)(10s + 1)} \tag{32}$$

specifications while the stability is preserved from Nyquist's plot. The maximum choice of choosing the finite zeros of $K(s)$ is 2 $(n-1=2)$. We choose two zeros of compensator at $-0{\cdot}1$ and $-1{\cdot}0$, respectively. The transfer function consisting of the set of chosen zeros and the poles of $G_p(s)$ is shown in Fig. 12 (solid line). This loop transfer function is given by (36).

$$118{\cdot}53\frac{(10s+1)(s+1)}{(10s-1)(2s+1)(s/3+1)} \tag{36}$$

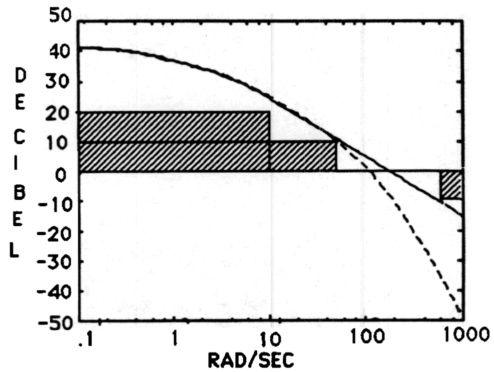

Figure 12. The Bode plot of $G_p(s)K(s)$ and the transfer function in (36).

Figure 13 shows the Nyquist plot of the loop transfer function in (36) for stability guarantee. Considering the zeros of the plant the compensator transfer function is given by (37).

$$K(s)=23{\cdot}71\frac{(10s+1)(s+1)}{(s/200+1)(s/300+1)(s/2+1)} \tag{37}$$

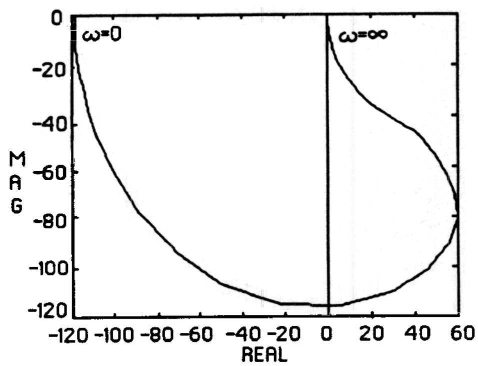

Figure 13. Nyquist's plot of transfer function given by (36).

The dashed plot in Fig. 12 shows the plot of $G_p(s)K(s)$ for the compensator designed above. It could be seen that the loop transfer function $G_p(s)K(s)$ satisfies the design specifications laid down by the engineer at lower and higher frequencies. By putting the poles of the compensator at the zeros of the plant, LTR was obtained and as shown in Figs. 12, 14 and 15. The solid line is (36) while the dashed line represents $G_p(s)K(s)$ in Figs. 12, 14 and 15.

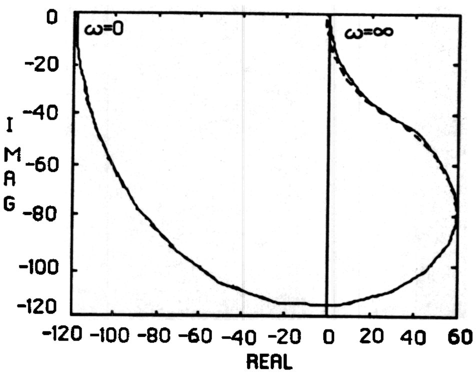

Figure 14. Nyquist's plot of $G_p(j\omega)K(j\omega)$ and the transfer function given by (36).

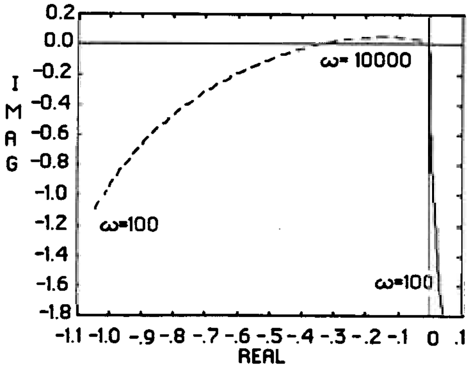

Figure 15. Nyquist's plot of $G_p(j\omega)K(j\omega)$ and the transfer function given by (36) at high frequencies.

6. Conclusion

In the LTR, the eigenvalues of the compensator $K(s)$ cancel the zeros of the plant. By exploring the eigenstructure of the LTR, when loop transfer recovery takes place, we provide a simple design procedure for SISO systems. The sufficient condition for LTR and the stability of the system is that the plant be minimum phase.

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